

# Equivariant Lorentzian Spectral Triples

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## Abstract

We present examples of equivariant noncommutative Lorentzian spectral geometries. The equivariance with respect to a compact isometry group (or quantum group) allows to construct the algebraic data of a version of spectral triple geometry adapted to the situation of an indefinite metric. The spectrum of the equivariant Dirac operator is calculated.

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# 1 Introduction

Equivariance under compact quantum groups has proven to be a very efficient tool to construct new explicit examples of spectral triples. The equivariance of the spectral triple algebra representation fixes the Hilbert space as a representation space of the quantum group and enables to diagonalize the Dirac operator on finite-dimensional subspaces invariant under its action. The eigenvalues of the Dirac can be, in turn, computed with the help of the order-one-condition.

In contrast to that situation, Lorentzian spectral triples, and in particular classical Lorentzian spin manifolds, generically admit *noncompact* isometry groups, with the relevant representations – occurring as eigenspaces of the Lorentzian Dirac operator – being infinite dimensional. From the point of view of a physicist this large space of solutions of the Dirac-equation (to a given mass) is of course desired. Moreover, as has been pointed out in [9], one may reconstruct the complete information about the metric and the spin structure of a (commutative) Lorentzian spin manifold from the space of solution to the Dirac-equation to a fixed mass. In that work it was also shown that certain Lorentzian spectral triples can, in principle, be constructed explicitly by exploiting their isometries.

However, the procedure adopted there appears much less systematic than the corresponding one for the Euclidean spectral triples. For instance, the decomposition of the Hilbert space into irreducible representations of the isometry group ( $SL(2, \mathbb{R})$  in that case) is not derived systematically but rather put in by hand, in order to circumvent some technical problems related to the infinite dimension of the relevant representations. Moreover, unlike in the case of compact isometry groups, not all series of representations appear in the decomposition of the Hilbert space.

Equivariant Lorentzian spectral triples are very interesting in a physical context for the model-building, especially since there are many known deformations of the Lorentz and Poincaré groups. From the mathematical point of view, equivariance is still the most efficient technical tool to construct the infinite-dimensional eigenspaces of the Dirac operator also in this situation. It appears, that this tool seems indispensable for the construction of genuine *noncommutative* examples of Lorentzian spectral triples.

As pointed out above, the main problem when dealing with equivariance in the Lorentzian case is the noncompactness of the full isometry groups. However, considering the few examples of compact Lorentzian manifolds,

which admit compact isometry groups, we can observe that these groups are necessarily much smaller than the isometry group of the same manifold when equipped with the (maximally symmetric) Riemannian metric.

As the construction of (Riemannian) spectral triples via equivariance uses the full isometry group, it may then seem at first sight that the (reduced) isometry groups for the Lorentzian case may not suffice for a systematic construction of a Lorentzian spectral triple.

In this paper we shall demonstrate that this is not the case, as the data of Lorentzian spectral triples involve an additional operator, the fundamental symmetry  $\beta$ . The equivariance condition for  $\beta$  together with that for  $J, D$  and the representation of  $\mathcal{A}$  indeed does provide enough equations to systematically construct a (compact) Lorentzian spectral triple (up to a scale) with classical, compact isometry groups.

The problem of constructing equivariant Lorentzian compact spectral triples can be in some cases reduced to the problem of finding a Euclidean spectral geometry but with an isometry group smaller than the maximally allowed. We shall see, that in some cases, like, for instance, in the  $SU_q(2)$  case, it is still not clear whether the full spectral data (including the reality structure) can be obtained.

## 2 Axioms for Lorentzian spectral triples

We begin with the axioms for real Lorentzian spectral geometries, which shall be discussed thoroughly in [13]. They are analogous to those for the Euclidean case [2] and, in fact, based on the idea that to each Lorentzian spectral geometry, it should be possible to associate a corresponding Euclidean one and thereby a well-defined index map [12].

**Definition 2.1.** A geometric real (odd or even) Lorentzian spectral triple of signature  $(1, q)$  is given by the data  $(\mathcal{A}, \pi, \mathcal{H}, D, J, \gamma, \beta)$ , where:

- $\mathcal{A}$  is an involutive algebra,  $\pi$  its faithful bounded star representation on a Hilbert space  $\mathcal{H}$ ,
- in the even case,  $1+q \in 2\mathbb{Z}$ ,  $\gamma = \gamma^\dagger$ ,  $\gamma^2 = 1$  is a  $\mathbb{Z}_2$  grading, commuting with the representation of  $\mathcal{A}$ ,
- $J$  is an antilinear isometry such that:

$$[J\pi(a)J^{-1}, \pi(b)] = 0, \quad \forall a, b \in \mathcal{A}, \quad (1)$$

- $\beta = -\beta^\dagger$ ,  $\beta^2 = -1$  is the  $\mathbb{Z}_2$ -grading associated with the Krein-space structure, commuting with the representation of the algebra  $\mathcal{A}$ ,
- $D$  an unbounded, densely defined operator, which is  $\beta$ -selfadjoint, that is:  $D^\dagger = \beta D \beta$ , and such that  $[D, \pi(a)]$  is bounded for every  $a \in \mathcal{A}$ , and  $D\gamma = -\gamma D$ .
- The operator

$$\langle D \rangle = \sqrt{\frac{1}{2}(DD^\dagger + D^\dagger D)} \quad (2)$$

has compact resolvent. In addition it is required that  $[\langle D \rangle, [D, \pi(a)]]$  is bounded for all  $a \in \mathcal{A}$ .

- The grading, reality structure and the Dirac operator satisfy:

$$DJ = \epsilon JD, \quad J^2 = \epsilon', \quad J\gamma = \epsilon''\gamma J. \quad (3)$$

where  $\epsilon, \epsilon', \epsilon''$  are  $\pm 1$  depending on  $1 - q$  modulo 8 according to the following rules:

$1 - q \bmod 8$	0	1	2	3	4	5	6	7
$\epsilon$	+	+	+	-	+	+	+	-
$\epsilon'$	+	+	+	-	-	-	-	+
$\epsilon''$	+		-		+		-	

- the Krein-space structure satisfies:

$$\beta\gamma = -\gamma\beta, \quad \beta J = -\epsilon^p J\beta, \quad (4)$$

- The Dirac operator satisfies the order-one condition:

$$[J\pi(a)J^{-1}, [D, \pi(b)]] = 0, \quad \forall a, b \in \mathcal{A}. \quad (5)$$

- There exists a Hochschild cycle of dimension  $n = 1 + q$ , valued in  $\mathcal{A}^o \otimes \mathcal{A}$ ,

$$c = a_{i_0}^o \otimes a_{i_o} \otimes a_{i_1} \otimes \cdots \otimes a_{i_n},$$

such that:

$$J\pi(a_{i_0})J^{-1}\pi(a_{i_1})[D, \pi(a_{i_1})] \cdots [D, \pi(a_{i_n})] = \begin{cases} \gamma & n \text{ even} \\ 1 & n \text{ odd} \end{cases},$$

- We say that the Lorentzian spectral triple has time-orientation if there exist  $a_i^o, a_i, b_i \in \mathcal{A}$  such that

$$\beta = \sum_i J\pi(a_i^o)J^{-1}\pi(a_i)[D, \pi(b_i)]. \quad (6)$$

If we do not assume existence of  $J$ , we have a spectral triple without real structure.<sup>1</sup>

We restrict ourselves only to the algebraic requirement and we refer the reader to the papers [13, 12] for details on further analytic requirements like summability, finiteness conditions as well as the Poincaré duality, which can be quite similarly postulated here, as it is done in the Euclidean case (see [4], for instance).

**Remark 2.2.** *The definition can be in a straightforward way extended to the case of arbitrary signature  $(p, q)$ . Our notation is that the Euclidean spectral geometry of dimension 0 is identified with the  $(0, n)$  signature.*

The sign relations for the arbitrary signature have been studied in the context of *spectral geometry axioms* by various people [7, 10, 8].

The basic motivation for the definition and the postulates for the algebraic formulation comes from the differential geometry of Lorentzian spin manifolds [1]. The Lorentzian version of the relations between the classical geometries and spectral geometries for appropriate commutative algebras is discussed elsewhere [13], we can quote:

**Lemma 2.3.** *(see [13] for details) Let  $M$  be a compact Lorentzian spin manifold. Then taking  $\mathcal{A} = C^\infty(M)$ ,  $\mathcal{H}$  to be the summable sections of the spinor bundle, and  $D$  the Dirac operator,  $(\mathcal{A}, \mathcal{H}, D)$  is a Lorentzian spectral triple.*

*The operator  $\beta$  could be identified with a specific choice of a fundamental symmetry (denoted  $J$  in [1]). Different, but normalized choices of  $\beta$  lead to equivalent Lorentzian spectral triples in the sense that for the choices  $\beta_1, \beta_2$  with  $\beta_i^2 = -1$  for  $i = 1, 2$ , there exists a unitary operator  $U : \mathcal{H}_1 \mapsto \mathcal{H}_2$  such that:*

$$X_2 = UX_1U^*, \quad X \in \{\beta, J, \gamma, \pi(\mathcal{A})\},$$

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<sup>1</sup>Note that in that case it is difficult to say whether we are in a typically Lorentzian case, with one "time-like" noncommutative direction. We might as well be in the case with a signature  $(p, q)$ ,  $p - q = 1 \pmod{2}$ .

and

$$[D_2, UaU^*] = U[D_1, \pi(a)]U^* \quad \forall a \in \mathcal{A}.$$

In this paper we discuss the genuine noncommutative examples.

## 2.1 Equivariance of Lorentzian triples

We recall the notion of equivariance for spectral triples [11, 15], extending it, in a natural way, to the Lorentzian case.

**Definition 2.4.** *Let  $H$  be a Hopf algebra, acting on the algebra  $\mathcal{A}$ . A Lorentzian spectral triple is  $H$ -equivariant if:*

- $H$  has a representation  $\rho$  on a dense domain of  $\mathcal{H}$ , such that the representation of  $\mathcal{A}$  is equivariant:

$$\rho(h)\pi(a) = \pi(h_{(1)} \triangleright a)\rho(h_{(2)}),$$

for all  $a \in \mathcal{A}, h \in H$ , with the equality valid on a dense domain of  $\mathcal{H}$ ,

- $\gamma, \beta$  and  $D$  commute with the representation  $\rho$  of  $H$
- the reality structure  $J$  is equivariant:

$$J(\rho(Sh)^\dagger)J^{-1} = \rho(h), \quad h \in H.$$

## 3 Lorentzian Spectral Triple for the Noncommutative Torus

The Euclidean spectral triple of the noncommutative torus is one of the best known examples. Also in the Lorentzian case it has been studied, albeit without the real structure in [16].

We shall investigate here it once again following exactly the procedure of equivariance [15], however, taking the Lorentzian axioms for  $(1, 1)$  spectral geometry.

### 3.1 Noncommutative Torus and its symmetries

We recall the basic definitions,

**Definition 3.1.** *Consider the Hilbert space  $l^2(\mathbb{Z}^2)$  with the orthonormal basis  $\{|n, m\rangle, n, m \in \mathbb{Z}\}$  and the unitary operators:*

$$\begin{aligned}\pi(U)|n, m\rangle &= |n+1, m\rangle, \\ \pi(V)|n, m\rangle &= \lambda^{-n}|n, m+1\rangle,\end{aligned}$$

where  $\lambda$  is complex number  $|\lambda| = 1$ . The algebra generated by these operators we shall call the algebra of functions on the noncommutative torus.

Note that we defined so far the algebra of polynomials and we might complete it either to a Fréchet algebra or a  $C^*$  algebra.

**Proposition 3.2.** *Let  $\mathfrak{u}(\mathbf{1}) \oplus \mathfrak{u}(\mathbf{1})$  be the Lie algebra generated by two derivations on the noncommutative torus:*

$$\begin{aligned}\delta_1 \triangleright U &= U, & \delta_2 \triangleright U &= 0, \\ \delta_1 \triangleright V &= 0, & \delta_2 \triangleright V &= V.\end{aligned}$$

Then, with the representation:

$$\begin{aligned}\rho(\delta_1)|n, m\rangle &= n|n, m\rangle, \\ \rho(\delta_2)|n, m\rangle &= m|n, m\rangle.\end{aligned}$$

we have the representation of the cross-product algebra of the functions on the noncommutative torus by the symmetry algebra. (The latter being the universal enveloping algebra of  $\mathfrak{u}(\mathbf{1}) \oplus \mathfrak{u}(\mathbf{1})$ .) Here, we take as the dense subspace  $V$  the linear space spanned by the basis  $|n, m\rangle, n, m \in \mathbb{Z}$ .

To construct the real Lorentzian spectral triple we need a grading  $\gamma$  (which just doubles the Hilbert space) the antilinear isometry  $J$ , and the Krein-space structure  $\beta$ .

Note that for  $(1, 1)$  we have  $\beta\gamma = -\gamma\beta$ , so by choosing the Hilbert space to be  $\mathcal{H} \otimes \mathbb{C}^2$  with the diagonal representation  $\text{diag}(\pi)$  and  $\gamma$  diagonal with  $\pm 1$ :

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

we have

$$\beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Next, using the modular operator from the Tomita-Takesaki theory,

$$J_0|n, m\rangle = \lambda^{-nm}| -n, -m\rangle,$$

we obtain  $J^2$ , by tensoring  $J_0$  with a suitable matrix from  $M_2(\mathbb{C})$ . To satisfy the algebraic requirements of the real  $(1, 1)$  spectral triple, we need to have:

$$J^2 = 1, \quad J\gamma = \gamma J, \quad J\beta = -\beta J.$$

so it is clear that  $J = J_0 \otimes \gamma$ : Taking  $\gamma$  to be block diagonal we have:

$$J = \begin{pmatrix} J_0 & 0 \\ 0 & -J_0 \end{pmatrix}. \quad (7)$$

Next, let us come to the point of constructing the equivariant Dirac operator, here again we repeat the steps from the Euclidean case.

Since  $D$  anticommutes with  $\gamma$ , it must be of the form

$$D = \begin{pmatrix} 0 & \partial_- \\ \partial_+ & 0 \end{pmatrix}.$$

Taking into account that  $D^\dagger = \beta D \beta$  is selfadjoint we have:

$$(\partial_\pm)^\dagger = \partial_\pm. \quad (8)$$

**Proposition 3.3.** *Every Dirac operator  $D$ , which is  $\mathfrak{u}(\mathbf{1}) \oplus \mathfrak{u}(\mathbf{1})$ -equivariant must be of the form given above, with  $\partial_\pm$ :*

$$\partial_\pm|n, m, \pm\rangle = d_{n,m}^\pm|n, m, \mp\rangle, \quad n, m \in \mathbb{Z}.$$

This follows directly from the requirement  $[D, \delta_i] = 0$ ,  $i = 1, 2$ . As we shall see, this assumption, together with other algebraic requirements fixes  $\partial$  up to a normalization factor.

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<sup>2</sup>Thus, we restrict ourselves to one choice of the spin structure on the noncommutative torus, for details see [14]



**Lemma 3.4.** *Any Dirac operator  $D$ , which has  $\mathfrak{u}(1) \oplus \mathfrak{u}(1)$  as an isometry and which is order-one (see (5)) on the  $(1,1)$  spectral geometry of the noncommutative torus, is defined by the set of real coefficients  $d_{n,m}^\pm$ :*

$$d_{n,m}^\pm = \tau_1^\pm n + \tau_2^\pm m + \epsilon, \quad (9)$$

*Proof.* First of all, using  $JD = DJ$  we immediately get that the coefficients  $d_{m,n}^\pm$  must satisfy:

$$(d_{m,n}^\pm)^* = -d_{-m,-n}^\pm. \quad (10)$$

Then, from order-one condition we get:

$$d_{n+1,m}^\pm = 2d_{n,m}^\pm - d_{n-1,m}^\pm, \quad (11)$$

$$d_{n+1,m}^\pm - d_{n,m}^\pm = d_{n+1,m-1}^\pm - d_{n,m-1}^\pm, \quad (12)$$

$$d_{n,m+1}^\pm - d_{n,m}^\pm = d_{n-1,m+1}^\pm - d_{n-1,m}^\pm, \quad (13)$$

$$d_{n,m+1}^\pm = 2d_{n,m}^\pm - d_{n,m-1}^\pm \quad (14)$$

The above recursion relations have solutions:

$$d_{n,m}^\pm = \tau_1^\pm n + \tau_2^\pm m + \epsilon^\pm, \quad (15)$$

for arbitrary constants  $\tau_i, \epsilon$ . Using (10) we first see that:

$$(\tau_i^\pm)^* = \tau_i^\pm, \quad i = 1, 2, \quad (\epsilon^\pm)^* = -\epsilon^\pm.$$

On the other hand, using (8) we obtain:

$$(\tau_i^\pm)^* = \tau_i^\pm, \quad i = 1, 2, \quad (\epsilon^\pm)^* = \epsilon^\pm,$$

Therefore all  $\tau_i^\pm$  must be real and  $\epsilon^\pm = 0$ .  $\square$

We can now compute the operator  $\langle D \rangle$  defined in (2), which comes out as

$$\langle D \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\partial_+^2 + \partial_-^2} & 0 \\ 0 & \sqrt{\partial_+^2 + \partial_-^2} \end{pmatrix}.$$

We shall have to investigate next whether this operator has compact resolvent. Since  $\langle D \rangle$  is already diagonalized,

$$\langle D \rangle |n, m, \pm\rangle = \frac{1}{\sqrt{2}} \sqrt{(d_{n,m}^+)^2 + (d_{n,m}^-)^2} |n, m, \pm\rangle,$$

**Lemma 3.5.** *The operator  $\langle D \rangle$  has compact resolvent whenever*

$$\tau_1^+ \tau_2^- \neq \tau_2^+ \tau_1^-.$$

*Proof.* We can rewrite the eigenvalues of  $\langle D \rangle^2$  as a quadratic form restricted to the  $\mathbb{Z}^2$  lattice of vectors in  $\mathbb{R}^2$ . On the basis vectors the form is:

$$\begin{pmatrix} (\tau_1^+)^2 + (\tau_1^-)^2 & \tau_1^+ \tau_2^+ + \tau_1^- \tau_2^- \\ \tau_1^+ \tau_2^+ + \tau_1^- \tau_2^- & (\tau_2^+)^2 + (\tau_2^-)^2 \end{pmatrix}$$

The form certainly non-negative but it is strictly positive on non-zero vectors if and only if its determinant is positive. This leads to:

$$(\tau_1^+ \tau_2^- - \tau_1^- \tau_2^+)^2 > 0,$$

hence the above condition.

If  $\tau_1^+ \tau_2^- = \tau_1^- \tau_2^+$ , the eigenvalues of  $\langle D \rangle^2$  are:

$$((\tau_1^+)^2 + (\tau_1^-)^2) \left( n + \frac{\tau_1^+}{\tau_2^+} m \right)^2.$$

and it is clear that for any  $\epsilon > 0$  we can find infinitely many pairs  $(m, n)$  such that  $n + \frac{\tau_1^+}{\tau_2^+} m < \epsilon$ .

Therefore for  $\tau_1^+ \tau_2^- = \tau_1^- \tau_2^+$  the operator  $\langle D \rangle$  does not have a compact resolvent.  $\square$

Moreover, we have:

**Proposition 3.6.** *The axiom of time-orientation holds if and only if  $\tau_1^+ \tau_2^- \neq \tau_2^+ \tau_1^-$ .*

*In that case*

$$\beta = \frac{\tau_2^- + \tau_2^+}{\tau_1^+ \tau_2^- - \tau_2^+ \tau_1^-} U^\dagger[D, U] - \frac{\tau_1^- + \tau_1^+}{\tau_1^+ \tau_2^- - \tau_2^+ \tau_1^-} V^\dagger[D, V]$$

*Proof.* A simple computation shows that  $[D, U]V = \lambda V[D, U]$ , and likewise  $[D, U]U = U[D, U]$ . Similarly for  $[D, V]$ .

Accordingly, any one-form  $\omega$  can be written as  $\omega = a_U[D, U] + a_V[D, V]$  with  $a_U, a_V \in \mathcal{A}$ . We can therefore make the Ansatz  $\beta = a_U[D, U] + a_V[D, V]$ . Now, since  $\beta$  commutes with  $\delta_1, \delta_2$  we immediately infer that  $a_U = wU^\dagger$  and  $a_V = zV^\dagger$  with  $w, z \in \mathbb{R}$ . (Reality follows from  $\beta^\dagger = -\beta$ .)

The resulting linear equation for  $w, z$  then has the above solution, respectively it has no solution if  $\tau_1^+ \tau_2^- = \tau_2^+ \tau_1^-$ .  $\square$

**Remark 3.7.** *It might be very instructive to compute the metrics on the set of pure states over  $\mathcal{A}$  corresponding to the above four parameter family of Dirac-Operators. However, the space of pure states is explicitly known only in the commutative case,  $\lambda = 1$ , in which it is just the two-dimensional Torus  $T^2$ .  $U, V$  can then be identified with  $U = e^{i\varphi_1}, V = e^{i\varphi_2}$  for the usual “coordinates”  $\varphi_1, \varphi_2 \in [0, 2\pi]$ .*

*We therefore consider only this commutative case in the following. It is then possible to compute the metric as follows: We may write*

$$D = \Gamma_1 \delta_1 + \Gamma_2 \delta_2, \quad \Gamma_i = \begin{pmatrix} 0 & \tau_i^+ \\ \tau_i^- & 0 \end{pmatrix}, \quad i = 1, 2.$$

*With this definition the components  $g_{ij}$  of the metric with respect to the coordinates  $\varphi_1, \varphi_2$  are then given by the standard formula*

$$\Gamma_i \Gamma_j + \Gamma_j \Gamma_i = -2g_{ij} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

*(Note that  $\delta_i = i \frac{\partial}{\partial \varphi_i}$  in this case.) Here we obtain (combining the components  $g_{ij}$  to a two-by-two matrix  $g$ ),*

$$g = - \begin{pmatrix} \tau_1^+ \tau_1^- & \frac{1}{2}(\tau_2^+ \tau_1^- + \tau_1^+ \tau_2^-) \\ \frac{1}{2}(\tau_2^+ \tau_1^- + \tau_1^+ \tau_2^-) & \tau_2^+ \tau_2^- \end{pmatrix}.$$

*Thus, we immediately infer that (unless  $\tau_1^+ \tau_2^- = \tau_2^+ \tau_1^-$ )*

$$\det(g) = \tau_1^+ \tau_1^- \tau_2^+ \tau_2^- - \frac{1}{4}(\tau_2^+ \tau_1^- + \tau_1^+ \tau_2^-)^2 = -\frac{1}{4}(\tau_2^+ \tau_1^- - \tau_1^+ \tau_2^-)^2 < 0.$$

*Hence  $g$  always has one positive and one negative eigenvalue and thus it is always of Lorentzian signature.*

Let us now return to the case of generic  $\lambda$ .

**Proposition 3.8.**  *$[\langle D \rangle, [D, a]]$  is bounded for all  $a \in \mathcal{A}$ .*

We only sketch the proof:

*Proof.* First of all we observe that

$$[\langle D \rangle, [D, U]] = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \tau_1^+ [\sqrt{\partial_+^2 + \partial_-^2}, U] \\ \tau_1^- [\sqrt{\partial_+^2 + \partial_-^2}, U] & 0 \end{pmatrix}$$

and similarly for  $V$ . Due to the Leibniz rule for commutators it is sufficient to consider the generators of  $\mathcal{A}$ . The result then follows by induction for all  $a \in \mathcal{A}$ .

We therefore only need to prove the boundedness of  $[\sqrt{\partial_+^2 + \partial_-^2}, U]$  and  $[\sqrt{\partial_+^2 + \partial_-^2}, V]$ . Since  $\partial_\pm$  are derivations on the algebra, this is a well known fact, and we need not repeat the somewhat lengthy proof here. Readers interested to see it are referred to [4].  $\square$

**Proposition 3.9.** *The axiom of orientation is fulfilled, i.e.  $\gamma$  can be written as a two-form which is the image of a Hochschild-cycle under  $\pi$ .*

*Proof.* Note that  $\gamma = \beta\sigma$  where

$$\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\tau_2^- - \tau_2^+}{\tau_1^+ \tau_2^- - \tau_2^+ \tau_1^-} U^\dagger[D, U] + \frac{\tau_1^- - \tau_1^+}{\tau_1^+ \tau_2^- - \tau_2^+ \tau_1^-} V^\dagger[D, V].$$

Hence  $\gamma$  is a two-form. The computation that this two-form is the image of a Hochschild-cycle is lengthy but straightforward, and completely analogous to that in [4]. We therefor leave it to the reader.  $\square$

So we can finally claim

**Theorem 3.10.** *The above data provide, if  $\tau_1^+ \tau_2^- \neq \tau_2^+ \tau_1^-$ , irreducible  $\mathfrak{u}(1) \oplus \mathfrak{u}(1)$ -equivariant Lorentzian spectral triples over the noncommutative torus.*

**Remark 3.11.** *Similarly as in [14] we can describe the spectrum of the Lorentzian Dirac operator on the noncommutative torus with the choice of a different spin structure. We recall that the four different spin structures were labelled by two numbers,  $\sigma_\pm$  which could take values 0 or  $\frac{1}{2}$ . The above presented case is for  $\sigma_+ = \sigma_- = 0$ , however, the construction can be easily extended to the remaining situations. The Dirac operator, for each of the spin structure, is:*

$$D \begin{pmatrix} 0 & \tau_1^+(n + \sigma_+) + \tau_2^+(m + \sigma_-) \\ \tau_1^-(n + \sigma_+) + \tau_2^-(m + \sigma_-) & 0 \end{pmatrix},$$

with real parameters  $\tau_1^\pm, \tau_2^\pm$ .

## 4 The Lorentzian noncommutative isospectral 3-sphere

In the three-dimensional, apart from the obvious case of a three-dimensional torus we have another example of a compact Lorentzian geometry: sphere. We look with  $(1, 2)$  signature, so we look for the operators  $\beta, J, D$  satisfying:

$$\begin{aligned}\beta &= -\beta^\dagger, & \beta^2 &= -1, \\ J^2 &= 1, & J\beta &= \beta J, \\ D^\dagger &= \beta D \beta, & JD &= -DJ.\end{aligned}\tag{16}$$

### 4.1 The isospectral deformation of the sphere

Let  $\lambda$  be a complex number of module one, which is not a root of unity. The isospectral deformation of the three-sphere was first presented in [5], then in [6]. The general result for the spectral triple construction of Drinfeld-type twists as isospectral deformations and their equivariance was discussed in [15].

We use the description of the algebra of  $S_\lambda^3$  as generated by operators  $a, b$  and their hermitian conjugates, which act on the Hilbert space of square integrable functions on  $S^3$ . Using the basis, we have the explicit formulae:

$$\begin{aligned}\pi(a)|l, m, n\rangle = & \lambda^{\frac{1}{2}(m-n)} \left( \frac{\sqrt{l+1+m}\sqrt{l+n+1}}{\sqrt{2l+1}\sqrt{2l+2}} |l^+, m^+, n^-\rangle \right. \\ & \left. - \frac{\sqrt{l-m}\sqrt{l-n}}{\sqrt{2l}\sqrt{2l+1}} |l^-, m^+, n^-\rangle \right),\end{aligned}\tag{17}$$

$$\begin{aligned}\pi(b)|l, m, n\rangle = & \lambda^{-\frac{1}{2}(m+n)} \left( \frac{\sqrt{l+1+m}\sqrt{l-n+1}}{\sqrt{2l+1}\sqrt{2l+2}} |l^+, m^-, n^-\rangle \right. \\ & \left. + \frac{\sqrt{l-m}\sqrt{l+n}}{\sqrt{2l}\sqrt{2l+1}} |l^-, m^-, n^-\rangle \right),\end{aligned}\tag{18}$$

where  $l^\pm, m^\pm, n^\pm$  is a shortcut notation for  $l \pm \frac{1}{2}, m \pm \frac{1}{2}, n \pm \frac{1}{2}$ .

The representation give above is equivariant with respect to the Drinfeld twist of the  $\mathcal{U}(su(2)) \otimes \mathcal{U}(su(2))$  Hopf algebra. We shall, however, consider the spinorial representation of the algebra, which differs by the twisting by a two-dimensional representation in the second  $su(2)$ . So, in addition to

doubling the Hilbert space, we need to take into account that the second  $u(1)$  action is different. Therefore, the equivariant spinorial representation of  $S_\theta^3$  is diagonal but not just doubled:

$$\pi(x) = \begin{pmatrix} \pi_+(x) & 0 \\ 0 & \pi_-(x) \end{pmatrix}, \quad (19)$$

where  $\pi_\pm$  differ from  $\pi_0$  through the rescaling of the generators:

$$\begin{aligned} \pi_\pm(a) &= \lambda^{\pm\frac{1}{4}} \pi_0(a), \\ \pi_\pm(b) &= \lambda^{\mp\frac{1}{4}} \pi_0(b), \end{aligned}$$

In our case, to get the Lorentzian spectral geometry we cannot keep the entire symmetry as the isometry, and we need to reduce it to a smaller one, which shall be the Drinfeld twist of  $\mathcal{U}(u(1)) \otimes \mathcal{U}(su(2))$ . This, however, shall be used only when looking for the equivariant reality and the Dirac operators.

The Krein-space structure operator  $\beta$  commutes both with the symmetries and with the representation of the algebra, hence it must be diagonal:

$$\beta = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \quad (20)$$

The reality structure  $J$ , which satisfies the relations (16) must be off-diagonal:

$$J = \begin{pmatrix} 0 & J_0^- \\ J_0^+ & 0 \end{pmatrix}, \quad (21)$$

where  $J_0^\pm$  is a canonical equivariant antilinear map, which maps the algebra to its commutant:

$$J_0^\pm |l, m, n\rangle = i^{2(m+n)} |l, -m, -n\rangle. \quad (22)$$

Let us verify that  $J^2 = 1$ :

$$\begin{aligned} J^2 |l, m, n, \pm\rangle &= J_0^\mp (i^{2(m+n)} |l, -m, -n, \mp\rangle) \\ &= i^{-4(m+n)} |l, m, n, \pm\rangle = |l, m, n, \pm\rangle, \end{aligned}$$

where we have used that  $m + n$  is always integer.

Next, we begin looking for all equivariant operators  $D$ , which satisfy the order one condition. From equivariance with respect to the full right

symmetry and the left  $u(1)$  part we infer that the most general form of an equivariant operator is:

$$\begin{aligned} D|l, m, n, +\rangle &= d_{11}(l, m)|l, m, n, +\rangle + d_{21}(l, m)|l, m+1, n, -\rangle, \\ D|l, m, n, -\rangle &= d_{12}(l, m)|l, m-1, n, +\rangle + d_{22}(l, m)|l, m, n, -\rangle. \end{aligned} \quad (23)$$

From  $JD = DJ$  condition we read:

$$\begin{aligned} d_{11}(l, m)^* &= d_{22}(l, -m), \\ d_{21}(l, m)^* &= d_{12}(l, -m). \end{aligned} \quad (24)$$

For practical reasons it is more convenient to use the variables  $l \pm m$ , so, instead of writing  $d_{12}(l, m)$  we shall write  $d_{12}(l+m, l-m)$ .

Next, looking at  $[JaJ, [D, b]]$ , we first obtain:

$$\begin{aligned} 2\sqrt{l-m+1} d_{21}(l+m, l-m+1) &= \sqrt{l-m} d_{21}(l+m, l-m) \\ &+ \sqrt{l-m+2} d_{21}(l+m, l-m+2), \end{aligned} \quad (25)$$

which has the solution:

$$d_{21}(l+m, l-m) = \frac{R(l+m)}{\sqrt{l-m}} + S(l+m)\sqrt{l-m},$$

with arbitrary functions  $R, S$ .

Similarly, from one of the coefficient of  $[JaJ, [D, b]]$  we read:

$$2d_{11}(l-\frac{1}{2}, m+\frac{1}{2}) = d_{11}(l, m) + d_{11}(l-1, m+1),$$

which has the solution:

$$d_{11}(l, m) = R'(l+m) + (l-m) S'(l+m),$$

with arbitrary functions  $R', S'$ .

Putting back these solutions into the formulae for the coefficients and looking again at the order one condition we obtain relations:

$$S'(l+m) - S'(l+1+m) = 0,$$

hence  $S'(l+m) = S'_0 = \text{const.}$  and:

$$S(l+m)\sqrt{l+2+m} = S(l+1+m)\sqrt{l+1+m},$$

which has the solution:

$$S(l+m) = S_0 \sqrt{l+1+m},$$

with the arbitrary multiplicative constant  $S_0$ .

Further, the condition for  $R'(l+m)$  becomes:

$$2R'(l+m+1) = R'(l+m) + R'(l+m+2),$$

which leads to

$$R'(l+m) = R'_1 + R'_0(l+m),$$

with  $R'_0, R'_1$  constant.

On the other hand, for  $R$  we obtain:

$$\sqrt{l+m} R(l+m-1) = \sqrt{l+m+1} R(l+m),$$

which has a solution:

$$R(l+m) = \frac{R_0}{\sqrt{l+m+1}}.$$

Using this result and looking once again at the relations from the order-one condition we obtain that  $R'_0 = S'_0$  and  $R_0$  must vanish.

Hence the Dirac operator has the following form, when acting on vectors  $|l, m, n, \pm\rangle$ :

$$\begin{aligned} D|l, m, n, +\rangle &= iRm |l, m, n, -\rangle \\ &\quad + S\sqrt{l+1+m}\sqrt{l-m} |l, m+1, n, -\rangle, \\ D|l, m, n, -\rangle &= -iRm |l, m, n, +\rangle \\ &\quad + S^*\sqrt{l-m+1}\sqrt{l+m} |l, m-1, n, +\rangle, \end{aligned} \tag{26}$$

where  $S$  is arbitrary complex number and  $R$  is real. It is easy to verify that these restrictions arise from the conditions  $D^\dagger = \beta D \beta$  and  $JD = -DJ$ .

**Lemma 4.1.** *The spectrum of the Dirac operator is:*

$$\lambda(l, m) = -\frac{1}{2}iR \pm \frac{1}{2}\sqrt{|S|^2(l+\frac{1}{2})^2 - (|S|^2 + R^2)(m+\frac{1}{2})^2}.$$

*We have the following possibilities:*



- $R = 0$ : spectrum is real and symmetric:

$$\lambda(l, m) = \pm |S| \sqrt{(l - m)(l + m + 1)},$$

with 0 being an eigenvalue with infinite multiplicity;

- $S = 0$ : spectrum is pure imaginary:

$$\lambda(l, m) = -\frac{1}{2}iR(1 \pm (1 + 2m)),$$

- $S \neq 0, R \neq 0$ : spectrum might contains a pure imaginary part and a complex part with imaginary part lying on the  $-\frac{1}{2}iR$  axis.

The multiplicity of the eigenvalues depends on the ratio  $\frac{|S|}{R}$ , if this is irrational then all eigenvalues have multiplicity 1, whereas in the rational case some of the eigenvalues might occur multiple (but finite) number of times.

Direct calculations lead to the following result:

**Lemma 4.2.** *The selfadjoint operator  $\frac{1}{2}\langle D \rangle^2$  has spectrum:*

$$\text{Spec}(\langle D \rangle^2) = \{R^2(m + \frac{1}{2} \pm \frac{1}{2})^2 + |S|^2(l - m)(l + m + 1)\} \quad (27)$$

and has compact resolvent if and only if  $R|S| \neq 0$ .

*Proof.* We restrict ourselves only to one set of eigenvalues. Taking  $m' = m + \frac{1}{2}$  and  $l' = l + \frac{1}{2}$  we can rewrite the formula:

$$\lambda_{\langle D \rangle^2} = R^2(m' + \frac{1}{2})^2 + |S|^2(l'^2 - m'^2).$$

Then, it is clear that the number of such eigenvalues, which are less than  $\Lambda$  is finite. Indeed, both components must be less than  $\Lambda$  since  $l'^2 \geq m'^2$ , so:

$$(m' + \frac{1}{2}) < \frac{\Lambda}{R}, \quad l'^2 < \frac{\Lambda}{|S|} + m'^2,$$

so, unless  $R = 0$  or  $|S| = 0$  we get an estimate on  $l$ , and therefore for each  $\Lambda$  only finite number of eigenvalues are below it.  $\square$

**Lemma 4.3.** *The following cycle:*

$$c = \sum_i c_i \otimes c'_i = \frac{i}{R} (a \otimes A + b \otimes B - A \otimes a - B \otimes b)$$

*gives the time-orientability:*

$$\beta = \sum \pi(c_i)[D, \pi(c'_i)].$$

We can summarize our result:

**Theorem 4.4.** *There exists a one-parameter family of  $su(2) \otimes_\theta u(1)$  equivariant Lorentzian spectral geometries on the noncommutative three sphere  $S_\theta^3$ , given by the Dirac operator 26.*

**Remark 4.5.** *It is worth noting that taking the  $\lambda = 1$  limit one recovers the construction (albeit in a different framework) and eigenvalues of the Lorentzian Dirac operator on the sphere, presented in [1].*

## 4.2 The reconstruction of the metric

Having the explicit form of the Dirac operator we might attempt to calculate the metric components, to see whether the obtained Lorentzian structure on the three-sphere has no singularities. Clearly, for the same reason as in the case of noncommutative torus, we are limited to the  $\lambda = 1$  example.

It is convenient to use the basis of the left-invariant (hermitian) one-forms on the 3-sphere:

$$\begin{aligned}\omega^1 &= b^* da - adb^* + bda^* - a^* db, \\ \omega^2 &= i(ad b - bda - a^* db^* + b^* da^*), \\ \omega^3 &= i(bdb^* + a^* da),\end{aligned}$$

then the inverse of the metric, calculated as

$$g^{ij} = \frac{1}{2} (\pi(\omega^i)\pi(\omega^j) + \pi(\omega^j)\pi(\omega^i)),$$

with  $\pi(x dy) = \pi(x)[D, \pi(y)]$ , becomes:

$$g_{ij} = \frac{1}{2} \begin{pmatrix} -|S|^2 & 0 & 0 \\ 0 & -|S|^2 & 0 \\ 0 & 0 & \frac{1}{4}R^2 \end{pmatrix}$$

and is exactly the constant Berger-type metric with signature  $(1, 2)$  (the Euclidean part has negative sign), which was the starting point of Helga Baum's approach [1].

## 5 Isospectral deformations

In the previous sections we have derived all equivariant Lorentzian geometries on two examples of noncommutative manifold: the noncommutative torus and the noncommutative three-sphere. In both examples we have obtained the *isospectral deformation*, that is the Dirac operator came out as the classical one. This should not be surprising, as one can easily generalize the theorem for *isospectral deformations* of [5] to the Lorentzian case, and we can claim:

**Proposition 5.1** (compare [5] Theorem 6). *Let  $M$  be Lorentzian spin manifold and  $(C^\infty(M), \mathcal{H}, D, \gamma, J, \beta)$  be the ingredients of the associated Lorentzian spectral triple. We assume that the isometry group of  $M$  has rank at least 2. Then  $M$  admits a natural one-parameter isospectral deformation  $M_\theta$ .*

*Proof.* All steps of the construction from [5], section 5, can be repeated in the Lorentzian case. Since  $U(1) \times U(1)$  is a subgroup of the isometry group,  $D$  and  $\gamma$  (in the even case) and the Krein-space structure operator  $\beta$  commute with the action of this group. Hence, if we take for the deformed spectral triple the same Krein-space structure  $\beta$ , and the same Dirac operator  $D$ , we retain all relations and properties of the triple with the exception of the orientability and the time-orientability axioms.  $\square$

**Remark 5.2.** *The orientability and time-orientability cannot be automatically extracted from the commutative spectral triple data. As an example one can take exactly  $S_\theta^3$ : in the classical case ( $\theta = 0$ ) the cochain giving the orientability axiom might be chosen as*

$$a \otimes a^* + b \otimes b^*,$$

*however this particular choice does not give a suitable orientability for the deformation  $S_\theta^3$ .*

## 6 The Lorentzian quantum sphere $SU_q(2)$

The  $\theta$ -deformations are not the only one existing deformations of the 3-sphere. In the Euclidean version the quantum deformation of  $SU_q(2)$  and the related spectral geometry [3] were one of the first examples of genuine noncommutative spectral geometries beyond the isospectral examples.

We shall study here the possibility of obtaining a Lorentzian-type spectral geometry for the  $SU_q(2)$ , seen as  $S_q^3$ . Of course, the total symmetry of the quantum space  $\mathcal{U}_q(su(2)) \otimes \mathcal{U}_q(su(2))$  cannot be preserved, however, we might have a reduced  $\mathcal{U}(u(1)) \otimes \mathcal{U}_q(su(2))$  equivariance.

We construct the spectral triple using the spinorial equivariant representation used in [3]. Still, though the representation of  $SU_q(2)$  is  $\mathcal{U}_q(su(2)) \otimes \mathcal{U}_q(su(2))$  equivariant, the Dirac operator shall be only  $\mathcal{U}_q(su(2)) \otimes u(1)$  equivariant.

We briefly recall the fundamentals of the representation used in [3].

For  $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$ , with  $\mu = -j, \dots, j$  and  $n = -j - \frac{1}{2}, \dots, j + \frac{1}{2}$ , we compose the pair of spinors:

$$|j\mu n\rangle\rangle := \begin{pmatrix} |j\mu n\uparrow\rangle \\ |j\mu n\downarrow\rangle \end{pmatrix}, \quad (28)$$

with the convention that the lower component is zero when  $n = \pm(j + \frac{1}{2})$  or  $j = 0$ . Furthermore, a matrix with scalar entries,

$$A = \begin{pmatrix} A_{\uparrow\uparrow} & A_{\uparrow\downarrow} \\ A_{\downarrow\uparrow} & A_{\downarrow\downarrow} \end{pmatrix},$$

is understood to act on  $|j\mu n\rangle\rangle$  by the rule

$$\begin{aligned} A|j\mu n\uparrow\rangle &= A_{\uparrow\uparrow}|j\mu n\uparrow\rangle + A_{\downarrow\uparrow}|j\mu n\downarrow\rangle, \\ A|j\mu n\downarrow\rangle &= A_{\downarrow\downarrow}|j\mu n\downarrow\rangle + A_{\uparrow\downarrow}|j\mu n\uparrow\rangle. \end{aligned} \quad (29)$$

The representation  $\pi' := \pi \otimes \text{id}$  of  $\mathcal{A}$  is given by

$$\begin{aligned} \pi'(a) |j\mu n\rangle\rangle &= a_{j\mu n}^+ |j^+ \mu^+ n^+\rangle\rangle + a_{j\mu n}^- |j^- \mu^+ n^+\rangle\rangle, \\ \pi'(b) |j\mu n\rangle\rangle &= \tilde{b}_{j\mu n}^+ |j^+ \mu^+ n^-\rangle\rangle + \tilde{b}_{j\mu n}^- |j^- \mu^+ n^-\rangle\rangle, \\ \pi'(a^*) |j\mu n\rangle\rangle &= \tilde{a}_{j\mu n}^+ |j^+ \mu^- n^-\rangle\rangle + \tilde{a}_{j\mu n}^- |j^- \mu^- n^-\rangle\rangle, \\ \pi'(b^*) |j\mu n\rangle\rangle &= \tilde{b}_{j\mu n}^+ |j^+ \mu^- n^+\rangle\rangle + \tilde{b}_{j\mu n}^- |j^- \mu^- n^+\rangle\rangle, \end{aligned}$$

where  $\tilde{a}_{j\mu n}^\pm$  and  $\tilde{b}_{j\mu n}^\pm$  are, up to phase factors depending only on  $j$ , the

following triangular  $2 \times 2$  matrices:

$$\begin{aligned}\tilde{a}_{j\mu n}^+ &= q^{(\mu+n-\frac{1}{2})/2} [j+\mu+1]^{\frac{1}{2}} \begin{pmatrix} q^{-j-\frac{1}{2}} \frac{[j+n+\frac{3}{2}]^{1/2}}{[2j+2]} & 0 \\ q^{\frac{1}{2}} \frac{[j-n+\frac{1}{2}]^{1/2}}{[2j+1][2j+2]} & q^{-j} \frac{[j+n+\frac{1}{2}]^{1/2}}{[2j+1]} \end{pmatrix}, \\ \tilde{a}_{j\mu n}^- &= q^{(\mu+n-\frac{1}{2})/2} [j-\mu]^{\frac{1}{2}} \begin{pmatrix} q^{j+1} \frac{[j-n+\frac{1}{2}]^{1/2}}{[2j+1]} & -q^{\frac{1}{2}} \frac{[j+n+\frac{1}{2}]^{1/2}}{[2j][2j+1]} \\ 0 & q^{j+\frac{1}{2}} \frac{[j-n-\frac{1}{2}]^{1/2}}{[2j]} \end{pmatrix}, \\ \tilde{b}_{j\mu n}^+ &= q^{(\mu+n-\frac{1}{2})/2} [j+\mu+1]^{\frac{1}{2}} \begin{pmatrix} \frac{[j-n+\frac{3}{2}]^{1/2}}{[2j+2]} & 0 \\ -q^{-j-1} \frac{[j+n+\frac{1}{2}]^{1/2}}{[2j+1][2j+2]} & q^{-\frac{1}{2}} \frac{[j-n+\frac{1}{2}]^{1/2}}{[2j+1]} \end{pmatrix}, \\ \tilde{b}_{j\mu n}^- &= q^{(\mu+n-\frac{1}{2})/2} [j-\mu]^{\frac{1}{2}} \begin{pmatrix} -q^{-\frac{1}{2}} \frac{[j+n+\frac{1}{2}]^{1/2}}{[2j+1]} & -q^j \frac{[j-n+\frac{1}{2}]^{1/2}}{[2j][2j+1]} \\ 0 & -\frac{[j+n-\frac{1}{2}]^{1/2}}{[2j]} \end{pmatrix},\end{aligned}$$

and the remaining matrices are the hermitian conjugates

$$\tilde{a}_{j\mu n}^{*\pm} = (\tilde{a}_{j\pm\mu-n}^\mp)^\dagger, \quad \tilde{b}_{j\mu n}^{*\pm} = (\tilde{b}_{j\pm\mu-n}^\mp)^\dagger.$$

It is, however, convenient to use the approximate representation from [3], that is representation up to compact operators.

We have:

**Lemma 6.1.** *The operator*

$$\beta |j\mu n \uparrow\rangle = i |j\mu n \uparrow\rangle, \quad \beta |j\mu n \downarrow\rangle = -i |j\mu n \downarrow\rangle,$$

*commutes with the algebra up to compact operators and satisfies the requirements for the Krein-structure:  $\beta^2 = -i, \beta = -\beta^\dagger$ .*

and

**Lemma 6.2.** *The following operator defined for  $-j - \frac{1}{2} < n < j + \frac{1}{2}$ ,  $j > 0$*

as:

$$\begin{aligned}
D|j, m, n, \uparrow\rangle &= (ir_{\uparrow}j + iR_{\uparrow})|j, m, n, \uparrow\rangle \\
&\quad + iS(j + n + \frac{1}{2})q^{j-2n} \left( \frac{[j - n + \frac{1}{2}]}{[j + n + \frac{1}{2}]} \right)^{\frac{1}{2}} |j, m, n, \downarrow\rangle, \\
D|j, m, n, \downarrow\rangle &= (-ir_{\downarrow}j - iR_{\downarrow})|j, m, n, \downarrow\rangle \\
&\quad - iS(j + n + \frac{1}{2})q^{j-2n} \left( \frac{[j - n + \frac{1}{2}]}{[j + n + \frac{1}{2}]} \right)^{\frac{1}{2}} |j, m, n, \uparrow\rangle,
\end{aligned} \tag{30}$$

where  $R, r, S$  are real parameters, and on the remaining elements of the basis:

$$D|j, m, \pm(j + \frac{1}{2}), \uparrow\rangle = (r_{\uparrow}j)|j, m, \pm(j + \frac{1}{2}), \uparrow\rangle, \tag{31}$$

is  $\mathcal{U}_q(su(2)) \otimes u(1)$  invariant,  $\beta$ -self-adjoint and has bounded commutators with the algebra elements.

*Proof.* Clearly the diagonal part of  $D$  is the same (up to bounded corrections) as in [3], so we might restrict ourselves only to the off-diagonal part, which we call  $D_o$ . From the equivariance,  $D_o$  must have the form:

$$D_o|j, m, n, \uparrow\rangle = d_o(j, n)|j, m, n, \downarrow\rangle,$$

.

The commutator  $[D, \pi(a)]$  reads:

$$\begin{aligned}
|j, m, n, \uparrow\rangle &= (d_o(j^+, n^+)\tilde{a}_{jmn}^+ \uparrow\uparrow - d_o^*(j, n)\tilde{a}_{jmn}^+ \downarrow\downarrow)|j^+, m^+, n^+, \downarrow\rangle \\
&\quad + d_o(j^+, n^+)\tilde{a}_{jmn}^+ \downarrow\uparrow|j^+, m^+, n^+, \downarrow\rangle \\
&\quad + (d_o(j^-, n^+)\tilde{a}_{jmn}^- \uparrow\uparrow - d_o^*(j, n)\tilde{a}_{jmn}^- \downarrow\downarrow)|j^-, m^+, n^+, \downarrow\rangle \\
&\quad + d_o(j^-, n^+)\tilde{a}_{jmn}^- \downarrow\uparrow|j^+, m^+, n^+, \downarrow\rangle.
\end{aligned}$$

Using the approximate representation, we see that these relations lead to the requirement that the following expressions remain bounded:

$$q^{j+m}q^{j+n}(d_o(j - \frac{1}{2}, n + \frac{1}{2}) - d_o(j, n)),$$

and

$$\sqrt{1 - q^{2j+2m+2}} \left( \sqrt{1 - q^{2j+2n+3}} d_o(j + \frac{1}{2}, n + \frac{1}{2}) - d_o(j, n) \sqrt{1 - q^{2j+2n+2}} \right)$$

First, we observe that if  $d_o(j, n) = q^{j-n}c_o(j, n)$  and  $c_o(j, n)q^j$  is bounded then the first expression is certainly bounded (of the order  $q^{2j}$ ) and the second leads to the requirement that:

$$c_o(j + \frac{1}{2}, n + \frac{1}{2}) - c_o(j, n)$$

is bounded.

This, however, is possible if at most:

$$c_o(j, n) \sim a_o(j + n)f(j - n) + g(j - n),$$

where  $a_o$  is a constant and  $f(j - n)$  is bounded. As for the function  $g$  we need only to guarantee that its growth is not too fast, so that the other estimates are valid.

In particular we can choose  $f \equiv 1, g \equiv 0$ , so that  $d_o(j, n) = (j + n + \frac{1}{2})q^{j-n}$  or  $f \equiv 0, g(j - n) = j - n + \frac{1}{2}$ , however it is convenient to rewrite both expressions by perturbing slightly functions by a bounded component,

In fact we can show that both:

$$(j + n + \frac{1}{2})q^{j-2n} \frac{\sqrt{[j - n + \frac{1}{2}]}}{\sqrt{[j + n + \frac{1}{2}]}}$$

$$(j - n + \frac{1}{2})q^j \frac{\sqrt{[j + n + \frac{1}{2}]}}{\sqrt{[j - n + \frac{1}{2}]}}$$

satisfy the requirements.

Indeed, we have:

$$\sqrt{\frac{[j - n + \frac{1}{2}]}{[j + n + \frac{1}{2}]}} = q^n \frac{\sqrt{1 - q^{2j-2n+1}}}{\sqrt{1 - q^{2j+2n+1}}}$$

so that:

$$(j + n + \frac{1}{2})q^{j-2n} \frac{\sqrt{[j - n + \frac{1}{2}]}}{\sqrt{[j + n + \frac{1}{2}]}} = q^{j-n} (j + n + \frac{1}{2}) \frac{\sqrt{1 - q^{2j-2n+1}}}{\sqrt{1 - q^{2j+2n+1}}}$$

and

$$(j - n + \frac{1}{2})q^j \frac{\sqrt{[j + n + \frac{1}{2}]}}{\sqrt{[j - n + \frac{1}{2}]}} = q^{j-n}(j - n + \frac{1}{2}) \frac{\sqrt{1 - q^{2j+2n+1}}}{\sqrt{1 - q^{2j-2n+1}}}$$

where it is clear that  $\sqrt{1 - q^{2j-2n+1}}$  and  $\sqrt{1 - q^{2j+2n+1}}$  are bounded functions and their proportion is also bounded.

In both cases in the  $q \rightarrow 1$  limit one recovers an  $su(2) \times u(1)$  invariant first-order differential operator. Out of these two possible terms only the first one is unbounded, the second has the form  $xq^{-x}$ , which is a bounded function and therefore could be considered rather as a small perturbation.  $\square$

For this reason, we shall rather concentrate our investigation on the other contribution, thus taking as  $D$  (30). Assuming (for simplicity)  $r_{\uparrow} = -r_{\downarrow}$  and  $R_{\uparrow} = R_{\downarrow} + r = \frac{3}{2}r$  we calculate the spectrum of  $D$ :

**Lemma 6.3.** *The spectrum of  $D$  is:*

$$\lambda_D = \frac{1}{2} \pm \sqrt{(-r^2(2j+1)^2 + S^2 q^{2j-4n}(j+n+\frac{1}{2})^2 \frac{[j-n+\frac{1}{2}]}{[j+n+\frac{1}{2}]})},$$

for  $-j \leq m \leq j$ ,  $-j - \frac{1}{2} < n < j + \frac{1}{2}$ ,  $j = 0, \frac{1}{2}, \dots$ , and

$$\lambda'_D = ir(2j + \frac{3}{2}),$$

if  $-j \leq m \leq j$ ,  $n = \pm(j + \frac{1}{2})$ ,  $j = 0, \frac{1}{2}, \dots$

We calculate spectrum of  $\langle D \rangle^2$ :

**Lemma 6.4.** *The operator  $\langle D \rangle^2$  has compact resolvent, and its approximate spectrum is:*

$$\lambda_{\langle D \rangle^2} = \frac{1}{2}r^2(j+1 \pm \frac{1}{2})^2 + S^2 q^{2(j-n)}(j+n+\frac{1}{2})^2 + o(q^j),$$

For this reason it is clear that  $\langle D \rangle^2$  has a compact resolvent, as the number of its eigenvalues smaller than any  $N > 0$  is always finite.



**Remark 6.5.** *Note that  $\beta$  commutes with the algebra only up to compact operators. This, however, is to be expected in the geometries, which arise from  $q$ -deformations. The classical ( $q \rightarrow 1$ ) limit yields a first-order differential operator, which gives the metric of a correct signature  $(1, 2)$  provided that  $|S|^2$  is bigger than  $\frac{1}{4}R^2$ , then, however,  $\beta$  still does not commute with the algebra.*

## 7 Conclusions and Outlook

In this paper we have shown that equivariance may be used to construct explicit examples of Lorentzian spectral triples for classical manifolds and their isospectral deformations.

In the  $q$ -deformed case, we have been able to use the equivariance to construct an unbounded Fredholm module, with  $D$  and elements of  $\mathcal{A}$  having bounded commutator, but, so far, we fail to establish the order-one condition (even up to compact operators). However, it seems to us, that the problem to find a suitable *order-one*  $D$  obeying is related to the reduction of the “isometry group” from  $\mathcal{U}_q(su(2)) \otimes \mathcal{U}_q(su(2))$  as used in [3] to  $\mathcal{U}_q(su(2)) \otimes u(1)$ . This itself, is an intriguing problem and we shall investigate it in details (in an Euclidean setup) in a forthcoming paper.

Nevertheless, one of the important reasons for presenting the  $SU_q(2)$  is the existence of the fundamental equivariant symmetry  $\beta$ , which can only be chosen to commute with the algebra up to compact operators and has no apparent classical limit. The assumption that the commutation relations could be relaxed appears a very natural choice, at least for  $q$ -deformed geometries. Still, the orientability axioms – full in the Euclidean case and additionally time-orientability in the Lorentzian case are a puzzle in this example. The nonexistence of the evident classical limit in the presented construction might suggest that going out of the Euclidean setup could open even more possibilities for geometries (even in the rough meaning of unbounded Fredholm modules) than expected.

Of course, if one wants to construct new examples of noncommutative Lorentzian spectral triples which may serve as candidates for models of spacetime, so that, in particular, the eigenvalues of  $D$  have infinite degeneracy, then one has to extend the application of equivariance to locally compact quantum groups. This, and in particular, the cases which have the  $q$ -deformed Lorentz and Poincaré symmetries, or with the renowned  $\kappa$ -

deformation of the Poincaré group, remains an important challenge for future work.

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